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Stability of the Stewartson layer in a rapidly rotating gas

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The linear stability of the Stewartson layer in a compressible fluid is studied. The viscosity and the heat conductivity are shown to be negligible for a special kind of infinitesimal disturbance. The basic equations of the disturbance are shown to reduce to those for a Boussinesq fluid subject to a virtual radial stratification. A Miles-type sufficient condition for stability and a Howard-type semicircle theorem are derived. The growth rates of unstable modes with wavenumber and shear strength are summarized in stability diagrams for typical cases. The results clarify the situation in which the stability of the Stewartson layer is governed by a balance between the shear strength and the temperature stratification in the layer.

1. Introduction and summary

Let us consider a compressible fluid which fills a rotating cylinder. The cylinder is divided into inner and outer parts by infinitesimal gaps at a certain radius in the end plates. The inner and outer parts of the end plates rotate with different angular velocities. The difference in the angular velocities is small with respect to their average. This is a situation in which a vertical Stewartson layer appears as is shown schematically in figure 1 (Stewartson 1957). What is the stability of this Stewartson layer? What kind of new effects arise owing to the compressibility of the fluid? This is the problem which we want to discuss in this paper.

For the incompressible case, an experimental study of Stewartson-layer instability was performed by Hide & Titman (1967). Theoretical work was done by Busse (1968), Siegmann (1974) and Hashimoto (1976). As far as the author is aware, however, compressible cases have not yet been studied. Studies of compressible cases are important not only in themselves but also in relation to gas centrifuges used for the enrichment of uranium.

Before starting the discussion of our stability problem, we summarize physically important aspects of our problem. In the incompressible case, the main bodies of fluid bounded by the inner and outer parts of the cylinder end plates rotate rigidly with these respective parts. The jump in the angular velocity is smoothed out by an outer layer of thickness $E^{\frac{1}{4}}$, where E is the Ekman number [defined in equation (2.3)]. The structure of this layer is determined by its Ekman extensions on the end plates. The inner layer is of thickness $E^{\frac{1}{3}}$ and plays the role of rechannelling secondary meridional circulation in the outer layer. The main features of these properties do not change in the compressible case (Matsuda & Hashimoto 1976, 1978; Matsuda & Takeda 1978). Expansions and contractions of the fluid particles along with their meridional motions cause the absorption or release of heat. This affects not only the temperature and the angular velocity but also the meridional circulation itself. The

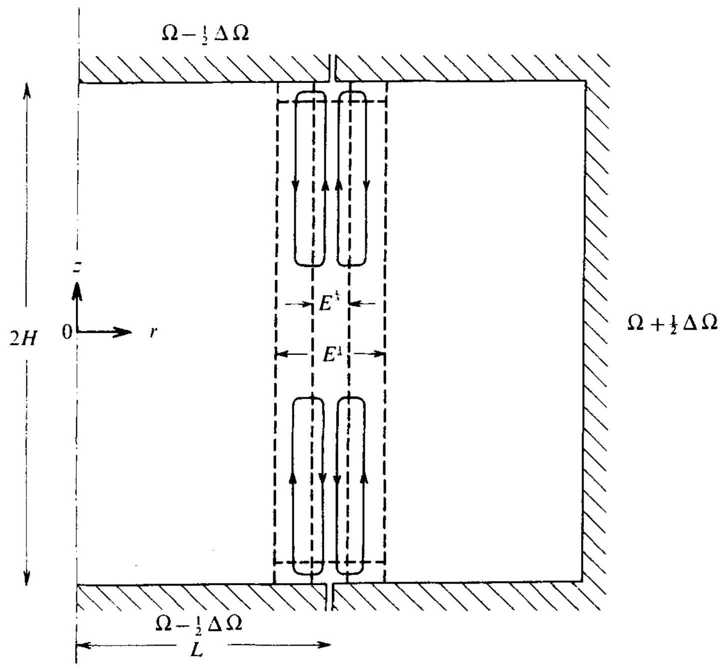

FIGURE 1. Schematic representation of the Stewartson layer and its Ekman extensions.

angular velocity is linearly correlated with the temperature by a thermal-wind relation. This reflects a dynamical balance between the Coriolis and the centrifugal forces. The meridional component of the velocity in the outer layer, however, is infinitesimally small in comparison with the azimuthal component (our frame of reference is a rotating system of co-ordinates with the mean angular velocity of the cylinder). Furthermore, if we restrict ourselves to the case in which the thickness of the outer layer is infinitesimally small in comparison with the radial scale height, the induced pressure is infinitesimal in comparison with the induced temperature and density. In the analysis of the instability of the outer layer, therefore, we neglect the effects of these small quantities. The inner layer is of negligibly small thickness in comparison with the outer layer and plays the minor role of rechannelling the secondary meridional circulation. The inner layer can be treated as a thin interface. Because the main configuration of the outer layer does not depend on the axial co-ordinate, the above restrictions reduce the outer Stewartson layer to a one-dimensional stratified shear layer subject to the centrifugal force field. The stratification is decomposed into two components: the background component, which corresponds to the density stratification related to the basic rigid-body rotation, and the induced component, which comes from the shear via the thermal-wind relation. The background component has a stabilizing effect. The induced component has a stabilizing or destabilizing effect according to whether the induced temperature decreases or increases in the radial direction. Because the centrifugal force plays the role of gravity in our problem, this trend can be understood as a Rayleigh–Taylor type of instability.

Our results are summarized in figures 2–4. The parameter β in these figures is a measure of the shear strength. A positive (negative) β represents a monotonic increase

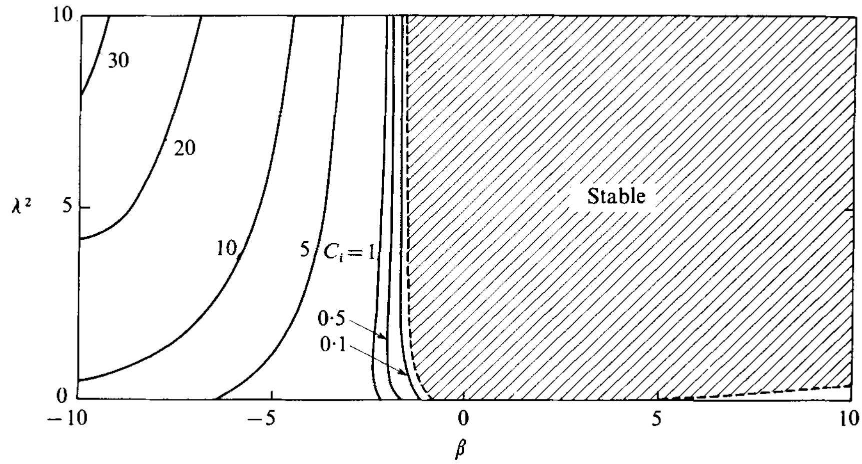

FIGURE 2. Stability diagram for a case with $Q_1 = Q_2 = 1$. The hatched region is the stable region corresponding to (3.4). The dashed lines are not neutral curves.

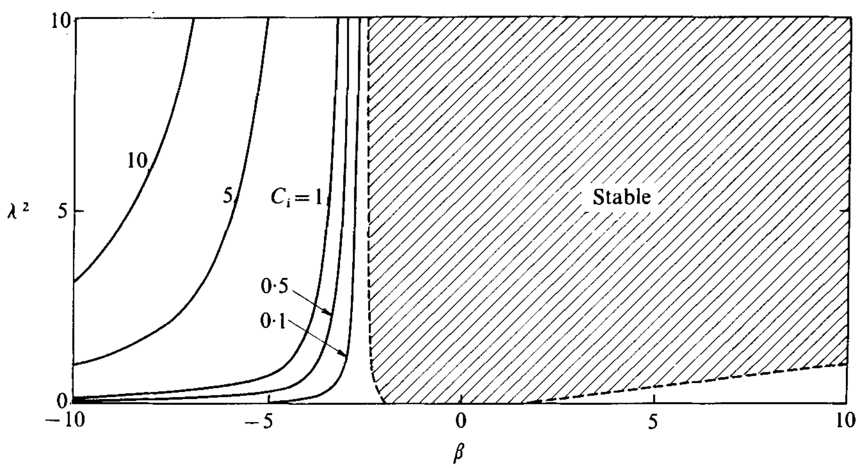

FIGURE 3. Stability diagram for a case with $Q_1 = 1$ and $Q_2 = 0$. Notation as in figure 2.

(decrease) in the azimuthal velocity in the radial direction. The parameter Q_1 is a measure of the basic density gradient and βQ_2 is a measure of the induced temperature gradient. From the thermal-wind relation, a positive (negative) β corresponds to an induced temperature stratification which is statically stable (unstable). On the left halves of figures 2 and 3, the effects of the induced temperature stratification and the shear co-operate with each other to induce instability. Thus no stable islands appear in these unstable halves. The penetration of the stable region into these left halves is ascribed to the stabilizing effects of the basic density stratification. On the right halves of figures 2 and 3, the stabilizing effect of the induced temperature stratification counteracts the destabilizing effect of the shear. The former is assisted by the stabilizing effect of the basic density stratification and causes wide extension of the stable region in the right halves of these figures. As the absolute value of β increases, both of the above effects increase. If the Prandtl number of the gas is sufficiently small, however,

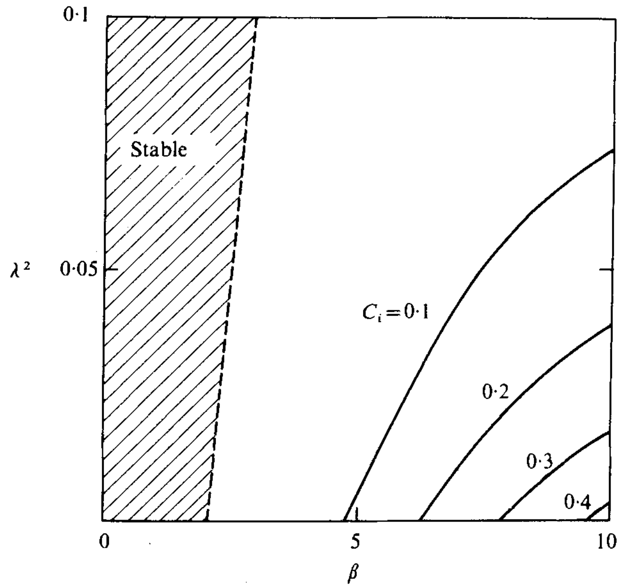

FIGURE 4. Magnification of the part $0 \leqslant \beta \leqslant 10$ and $0 \leqslant \lambda^2 \leqslant 0{\cdot}1$ of figure 3.

the increase in the destabilizing effect of the shear overwhelms that in the stabilizing effect of the induced temperature stratification. Thus an unstable island appears at the far right in the right half of figure 3 (this part is shown magnified in figure 4).

In § 2, we discuss the basic equations. In § 3, we derive a Miles-type sufficient condition for stability and a Howard-type semicircle theorem. In § 4, we give the method of solution of the eigenvalue problem.

2. Basic equations

Let us start from rigid-body rotation of a gas with uniform temperature T_0, which is confined in a cylinder. The cylinder is of height $2H$ and rotates around a vertical (z) axis with angular velocity Ω. If there exist narrow gaps at a radius L in each end plate and the inner and outer parts of the cylinder rotate with angular velocity $\Omega \mp \frac{1}{2}\Delta\Omega$, respectively, a Stewartson layer appears to smooth out the jump in the angular velocity (see figure 1). Because of the coupling between the thermodynamical and dynamical quantities, it is difficult to obtain the overall solutions even in the simple configuration shown in figure 1. However, Matsuda & Hashimoto (1976, 1978) and Matsuda & Takeda (1978) showed that the outer Stewartson layer of thickness $E^{\frac{1}{4}}$ has a structure similar to that in an incompressible fluid. As the unperturbed state of the gas, we take the $E^{\frac{1}{4}}$-layer configuration superposed on a basic state of rigid-body rotation: we introduce a rotating system of cylindrical co-ordinates (r, θ, z) which rotates around the z axis. The angular velocity of the system is Ω and the origin of the system is the mid-point of the z axis. The unperturbed state, in this rotating system, is

$$\left.\begin{array}{l} T_u = 1 + R_0 \overline{T}, \quad \rho_u = \epsilon_B(1 + R_0 \overline{\rho}), \quad p_u = \epsilon_B(1 + R_0 \overline{p}), \\ u_u = \overline{u}, \quad v_u = \overline{v}, \quad w_u = \overline{w}, \end{array}\right\} \tag{2.1}$$

where $\epsilon_B = \exp(\frac{1}{2} G_0 r^2), \quad G_0 = \Omega^2 H^2 / R T_0, \quad R_0 = |\Delta\Omega| / \Omega,$

suffixes u and B refer to the unperturbed and to the basic state, respectively, and bars refer to the $E^{\frac{1}{4}}$-layer. The solutions corresponding to the $E^{\frac{1}{4}}$-layer were given by Matsusa & Hashimoto. These are

$$\bar{T} = -\frac{\Gamma-1}{2\Gamma} r_0 Pr\, \bar{v}, \quad \bar{\rho} = -\bar{T}, \quad \bar{v} = \alpha \operatorname{sgn}(\eta)(1-e^{-\sigma|\eta|}), \tag{2.2a–c}$$

$$\bar{u} = O(E^{\frac{1}{4}}), \quad \bar{w} = O(E^{\frac{1}{4}}), \quad \bar{p} = O(E^{\frac{1}{4}}), \tag{2.2d–f}$$

where

$$\left.\begin{aligned} \eta &= (r-r_0)E^{-\frac{1}{4}}, \quad \sigma = \exp\left(\tfrac{1}{8}G_0 r^2\right)\left(1+\frac{\Gamma-1}{4\Gamma} Pr\, G_0 r_0^2\right)^{\frac{1}{4}}, \\ E &= \nu/H^2\Omega, \quad Pr = \nu/\kappa. \end{aligned}\right\} \tag{2.3}$$

In the above equations we use non-dimensional quantities in which the temperature T is non-dimensionalized by the uniform temperature T_0 of the basic state, the density ρ and the pressure p by their respective values at the origin, both the radial and the axial co-ordinate r and z by the half-distance H between the end plates and the velocity $(\bar{u}, \bar{v}, \bar{w})$ by $|\Delta\Omega| H/G_0$. The quantity η is a stretched radial co-ordinate in the $E^{\frac{1}{4}}$-layer, r_0 the non-dimensional radius of the $E^{\frac{1}{4}}$-layer, Γ the ratio of the specific heats, ν the kinematic viscosity, κ the thermometric conductivity and R the gas constant. The non-dimensional parameters G_0, Pr and E are the Mach number squared, the Prandtl number and the Ekman number. Because our system rotates rapidly, G_0 is of order unity and E and R_0 are infinitesimally small. The Prandtl number is of order unity. The parameter α is a measure of the shear strength in the $E^{\frac{1}{4}}$-layer.

Let us consider small disturbances $\mathbf{v}'(u', v', w')$, p', ρ' and T' to the unperturbed state $\bar{\mathbf{v}}(\bar{u}, \bar{v}, \bar{w})$, p_u, ρ_u and T_u. The linearized basic equations of the disturbances are

$$\frac{G_0}{\delta}\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{v} + \mathbf{F}\cdot\mathbf{v} + R_0\{\bar{\rho}\mathbf{F}\cdot\mathbf{v} + \rho\mathbf{F}\cdot\bar{\mathbf{v}} + \nabla\cdot\bar{\rho}\mathbf{v} + \nabla\cdot\rho\bar{\mathbf{v}}\} = 0, \tag{2.4}$$

$$\begin{aligned} \frac{1+R_0\bar{\rho}}{\delta}\frac{\partial \mathbf{v}}{\partial t} + \frac{R_0}{G_0}\{(\bar{\mathbf{v}}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\bar{\mathbf{v}}\} + 2\mathbf{k}\times\mathbf{v} + 2R_0\mathbf{k}\times(\bar{\rho}\mathbf{v}+\rho\bar{\mathbf{v}}) + G_0\rho\mathbf{k} \\ \times(\mathbf{k}\times\mathbf{r}) = -p\mathbf{F} - \nabla p + (E/\epsilon_B)\{\tfrac{1}{3}\nabla(\nabla\cdot\mathbf{v}) + \Delta\mathbf{v}\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{1+R_0\bar{\rho}}{\delta}\frac{\partial T}{\partial t} + \frac{R_0}{G_0}\{(\bar{\mathbf{v}}\cdot\nabla)T + (\mathbf{v}\cdot\nabla)\bar{T}\} \\ + \frac{\Gamma-1}{G_0}\{\nabla\cdot\mathbf{v} + R_0(p\nabla\cdot\bar{\mathbf{v}} + \bar{p}\nabla\cdot\mathbf{v})\} = \frac{\Gamma E}{Pr\,\epsilon_B}\Delta T, \end{aligned} \tag{2.6}$$

$$p = \rho + T, \tag{2.7}$$

where $\mathbf{F} = (G_0 r, 0, 0)$, $\mathbf{k} = (0, 0, 1)$ and δ is a time scaling factor to be determined later. In the above equations, primes are omitted for the sake of simplicity. We neglect terms of second order with respect to the disturbances and the Rossby number. Because disturbances which are trapped within the $E^{\frac{1}{4}}$-layer are expected to be the most unstable, we restrict ourselves to this layer. Correspondingly, we introduce the following variables:

$$x = \sigma\eta = \sigma(r-r_0)E^{-\frac{1}{4}} \quad (-\infty < x < \infty), \quad y = r_0\theta \quad (0 \leqslant y \leqslant 2\pi r_0). \tag{2.8}$$

In accordance with this change of variables, we expand the physical disturbance quantities as

$$\begin{pmatrix} \mathbf{v} \\ p \\ \rho \\ T \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 \\ p_0 \\ \rho_0 \\ T_0 \end{pmatrix} + E^{\frac{1}{4}}\begin{pmatrix} \mathbf{v}_1 \\ p_1 \\ \rho_1 \\ T_1 \end{pmatrix} + E^{\frac{1}{2}}\begin{pmatrix} \mathbf{v}_2 \\ p_2 \\ \rho_2 \\ T_2 \end{pmatrix} + O(E^{\frac{3}{4}}). \tag{2.9}$$

The most important point in our analysis is our assumption that $\delta = E^{-\frac{1}{4}}$ and R_0 is of order $E^{\frac{1}{4}}$. Because of this assumption, the Coriolis force terms, the cross-interaction terms between disturbances and unperturbed configurations, and the inertia terms all become of comparable magnitude. Note that the present time scale $1/\Omega E^{\frac{1}{4}}$ is much greater than the local Brunt–Väisälä frequency $1/\Omega r_0 G_0^{\frac{1}{2}}$ of the basic density stratification. With the above choice of parameters, the relevant set of the equations for the disturbances is

$$\sigma\frac{\partial u_1}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z} = 0, \tag{2.10}$$

$$2v_0 + r_0 G_0 \rho_0 = \sigma\, \partial p_1/\partial x, \tag{2.11}$$

$$\frac{\partial v_0}{\partial t}+\frac{R_0}{G_0 E^{\frac{1}{4}}}\left[\bar{v}\frac{\partial v_0}{\partial y}+\sigma u_1\frac{d\bar{v}}{dx}\right]+2u_1 = -\frac{\partial p_1}{\partial y}, \tag{2.12}$$

$$\frac{\partial w_0}{\partial t}+\frac{R_0}{G_0 E^{\frac{1}{4}}}\left[\bar{v}\frac{\partial w_0}{\partial y}\right] = -\frac{\partial p_1}{\partial z}, \tag{2.13}$$

$$\frac{\partial T_0}{\partial t}+\frac{R_0}{G_0 E^{\frac{1}{4}}}\left[\bar{v}\frac{\partial T_0}{\partial y}+\sigma u_1\frac{d\bar{T}}{dx}\right]-(\Gamma-1)\left[\frac{\partial \rho_0}{\partial t}+r_0 u_1+\frac{R_0}{G_0 E^{\frac{1}{4}}}\left(\sigma u_1\frac{d\bar{\rho}}{dx}+\bar{v}\frac{\partial \rho_0}{\partial y}\right)\right] = 0, \tag{2.14}$$

$$\rho_0 + T_0 = 0. \tag{2.15}$$

To derive these equations, we used the fact that $u_0 = 0$ and $p_0 = 0$. This follows directly from the zeroth-order continuity equation and the radial component of the momentum equation. In the derivation of the third term of the energy equation (2.14), we have used the second-order continuity equation. It is interesting that the effects of viscosity and thermal conductivity on the disturbances disappear from our basic equations (2.10)–(2.15).

To clarify the meaning of the above ordering, we consider other orderings of the parameters R_0 and δ. When $R_0 < O(E^{\frac{1}{4}})$, the cross-interactions of the disturbances with the unperturbed configuration disappear. The resulting equations reduce to those of inertial waves in a rotating gas. To examine cases in which $R_0 > O(E^{\frac{1}{4}})$, let us assume that $R_0 = O(E^{\frac{1}{8}})$. In this case, we can take $\delta \sim E^{-\frac{1}{8}}$. The resulting equations reduce to a special case of our basic equations in which the Coriolis force term and the pressure term in (2.12), the pressure term in (2.13) and the term $-(\Gamma-1)r_0 u_1$ in (2.14) are absent.

Eliminating all variables except u_1 from (2.10)–(2.15) and applying a single-mode analysis in which u_1 is assumed to be of the form

$$u_1 = \mathrm{Re}\left[U(x)\exp\{i(my+nz-\omega t)\}\right], \tag{2.16}$$

we obtain a single differential equation

$$(\beta\bar{V}-C)^2\frac{d^2U}{dx^2}+\left\{2\lambda^2\left(2+\beta\frac{d\bar{V}}{dx}\right)+Q_1(1+\lambda^2)+Q_2(1+\lambda^2)\beta\frac{d\bar{V}}{dx}-(\beta\bar{V}-C)\beta\frac{d^2\bar{V}}{dx^2}\right\}U = 0, \tag{2.17}$$

where

$$\left.\begin{aligned}\beta = \sigma\alpha R_0/G_0 E^{\frac{1}{4}},\quad C = C_r + iC_i = \sigma\omega/m,\quad \lambda^2 = n^2/m^2,\\ Q_1 = \frac{\Gamma-1}{\Gamma}G_0 r_0^2,\quad Q_2 = \frac{\Gamma-1}{2\Gamma}G_0 r_0^2 Pr\end{aligned}\right\} \tag{2.18}$$

and

$$\bar{V} = \mathrm{sgn}(x)(1-e^{-|x|}),\quad \bar{T} = (-\alpha Q_2/G_0 r_0)\bar{V}. \qquad (2.19), (2.20)$$

For the sake of convenience, let us divide our x space into positive and negative parts and denote the corresponding $U(x)$ by $U_+(x)$ and $U_-(x)$. Here $U_+(x)$ and $U_-(x)$ tend to zero at plus and minus infinity respectively. Because the disturbances are inviscid and non-conducting, the matching conditions at $x = 0$ are that u_1 and p_1 are continuous:

$$U_+(0) = U_-(0), \quad U'_+(0) = U'_-(0). \tag{2.21}$$

A search for non-trivial solutions $U_\pm(x)$ subject to these conditions constitutes an eigenvalue problem.

Equation (2.17) is formally the same as that of small disturbances on a shear flow $\beta\bar{V}$ in a Boussinesq fluid in which the virtual density stratification $\tilde{\rho}(x)$ is given by

$$\left(\frac{H_0 g_0}{\bar{V}_0^2}\right)\frac{d}{dx}\left(\frac{\tilde{\rho}}{\rho_0}\right) = 2\lambda^2\left(2+\beta\frac{d\bar{V}}{dx}\right)+Q_1(1+\lambda^2)+Q_2(1+\lambda^2)\beta\frac{d\bar{V}}{dx}, \tag{2.22}$$

where V_0, H_0 and ρ_0 are typical values of the velocity, the length scale and the density, respectively, and g_0 is the virtual gravitational acceleration in the $+x$ direction. A situation in which $d\tilde{\rho}/dx > 0$ corresponds to a stable stratification. The first term on the right-hand side of (2.22) comes from the Coriolis force. The second term comes from the basic density stratification. The third term comes from the induced temperature stratification in the $E^{\frac{1}{4}}$-layer, which is expressed in terms of $\beta\bar{V}$ by the thermal-wind relation (2.2a). When $\beta > 0$, i.e. when $\bar{v}$ increases monotonically with x, these three terms are all positive. This gives us a stable virtual stratification. If β is fixed and λ is increased, the flow becomes more stable. If λ is fixed and β is increased, the virtual density gradient increases. At the same time, however, the shear becomes stronger. The stability in this case is determined by a balance between the destabilizing effect of the shear and the stabilizing effect of the virtual stratification. This leads to the appearance of an unstable island in the right half of figure 3. When $\beta < -2$, the first term becomes negative in a neighbourhood of $x = 0$ and this region (in x space) expands as β decreases. The second term is positive irrespective of β. This corresponds to a stabilizing basic density stratification and is the cause of penetration of the stable region into the region of negative β. The third term is negative and its absolute value increases as $|\beta|$ and λ increases. Thus the flow becomes more unstable if β decreases at a fixed value of λ.

3. Derivation of Miles- and Howard-type theorems

Following Howard's (1961) proof of Miles' (1961) theorem on the stability of heterogeneous shear flow, let us introduce a new variable $G = (\beta\bar{V}-C)^{-\frac{1}{2}}U$ into (2.17). Multiplying the resulting equation by the complex conjugate of G and integrating with respect to x, we obtain

$$\int_{-\infty}^{\infty}\Big[(\beta\bar{V}-C)|G'|^2+\tfrac{1}{2}\beta\bar{V}''|G|^2-(\beta\bar{V}-C^*) \times[-\tfrac{1}{4}(\beta\bar{V}')^2+4\lambda^2+Q_1(1+\lambda^2)+\{2\lambda^2+Q_2(1+\lambda^2)\}\beta\bar{V}']\left|\frac{G}{\beta\bar{V}-C}\right|^2\Big]\,dx = 0, \tag{3.1}$$

where C^* denotes the complex conjugate of C and the primes denote differentiation with respect to x. Because the solutions U_+ decrease exponentially as $|x|\to\infty$ (see

§ 4 for proof), the above integration converges for $C_i \neq 0$. Taking the imaginary part of (3.1), we obtain

$$C_i \int_{-\infty}^{\infty} \Big[|G'|^2 + [-\tfrac{1}{4}(\beta\bar{V}')^2 + 4\lambda^2 + Q_1(1+\lambda^2) + \{2\lambda^2 + Q_2(1+\lambda^2)\}\beta\bar{V}'] \left|\frac{G}{\beta\bar{V}-C}\right|^2 \Big] dx = 0. \quad (3.2)$$

Therefore a sufficient condition for stability is

$$-\tfrac{1}{4}(\beta\bar{V}')^2 + 4\lambda^2 + Q_1(1+\lambda^2) + \{2\lambda^2 + Q_2(1+\lambda^2)\}\beta\bar{V}' > 0. \quad (3.3)$$

Noting that $0 \leqslant \bar{V}' \leqslant 1$, we can rewrite (3.3) as

$$-\tfrac{1}{4}\beta^2 + Q_2(1+\lambda^2)\beta + 6\lambda^2 + Q_1(1+\lambda^2) > 0. \quad (3.4)$$

The region corresponding to (3.4) is hatched in the stability diagrams in figures 2–4.

Let us next introduce $F = (\beta\bar{V} - C)^{-1}U$ into (2.17). Multiplying the resulting equation by the complex conjugate of F and integrating with respect to x, we obtain

$$\int_{-\infty}^{\infty} -(\beta\bar{V}-C)^2|F'|^2 + [4\lambda^2 + Q_1(1+\lambda^2) + \{2\lambda^2 + Q_2(1+\lambda^2)\}\beta\bar{V}']|F|^2 dx = 0. \quad (3.5)$$

Taking the imaginary part of (3.5), we find

$$C_i \int_{-\infty}^{\infty} (\beta\bar{V} - C_r)|F'|^2 dx = 0. \quad (3.6)$$

This shows that C_r must lie in the range $(-|\beta|, |\beta|)$ for an unstable mode ($C_i > 0$). Taking the real part of (3.5) and using (3.6) and the fact that $|\bar{V}| < 1$, we obtain

$$(C_r^2 + C_i^2 - \beta^2)\int_{-\infty}^{\infty} |F'|^2 dx < -\int_{-\infty}^{\infty} [4\lambda^2 + Q_1(1+\lambda^2) + \{2\lambda^2 + Q_2(1+\lambda^2)\}\beta\bar{V}']|F|^2 dx. \quad (3.7)$$

This implies that
$$C_r^2 + C_i^2 < \beta^2, \quad (3.8)$$

provided that
$$\beta > -\frac{4\lambda^2 + Q_1(1+\lambda^2)}{2\lambda^2 + Q_2(1+\lambda^2)}. \quad (3.9)$$

This corresponds to Howard's (1961) semicircle theorem.

4. Method of solution

Let us consider an eigenvalue problem in which an eigenvalue C is to be determined for a set of values of the parameters Q_1, Q_2, β and λ^2. The basic equation is (2.17) and the boundary conditions are the matching conditions (2.21) and conditions at infinity. If we introduce a variable $\xi = e^x$, the equation (2.17) for U_- becomes

$$(\beta\xi - \beta - C)^2(\xi^2 d^2U_-/d\xi^2 + \xi dU_-/d\xi) + [-\beta^2\xi^2 + \{\beta + C + 2\lambda^2 + Q_2(1+\lambda^2)\}\beta\xi + 4\lambda^2 + Q_1(1+\lambda^2)]U_- = 0. \quad (4.1)$$

We assume an asymptotic solution of the form

$$U_- = \xi^\mu \sum_{j=0}^{\infty} a_j \xi^j. \quad (4.2)$$

Substituting (4.2) into (4.1) and equating coefficients of equal powers of ξ, we find

$$\{(\beta+C)^2(\mu+j)^2+4\lambda^2+Q_1(1+\lambda^2)\}a_j+\beta\{-2(\beta+C)(\mu+j-1)^2 +\beta+C+2\lambda^2+Q_2(1+\lambda^2)\}a_{j-1}+\beta^2\{(\mu+j-2)^2-1\}a_{j-2}=0, \quad (4.3)$$

where $a_{-1} = a_{-2} = 0$. The indicial equation for $j = 0$ gives us

$$\mu^2 = -\frac{4\lambda^2+Q_1(1+\lambda^2)}{(\beta+C)^2}, \quad \mathrm{Re}\,\mu > 0. \quad (4.4)$$

Iterative use of (4.3) shows that the coefficients a_j ($j \geqslant 1$) can be expressed in terms of a_0. Because of the linearity of (4.3), the resultant solution is of the form $U_- = a_0\overline{U}_-$. Equation (4.1) has a singular point at $\xi_s = 1+C/\beta$, so that the expression (4.2) does not converge at $\xi = 1$ ($x = 0$) if $|\xi_s| < 1$. To evaluate the values of $\overline{U}_-$ and $\overline{U}'_-$ at $\xi = 1$, we start from the asymptotic expression (4.2) at a certain point $\xi_0 < |\xi_s|$ and integrate (4.1) numerically by the Runge–Kutta–Gill method of quadrature. We calculate U_+ in a similar manner. Again the solution is of the form $U_+ = a_0\overline{U}_+$. Now that we have obtained $\overline{U}_+(0)$, $\overline{U}'_+(0)$, $\overline{U}_-(0)$ and $\overline{U}'_-(0)$, we can estimate $U_+(0)\,U'_-(0) - U'_+(0)\,U_-(0)$ for any complex value of C for a given set of values of the parameters λ^2, β, Q_1 and Q_2. We plot this estimate on the complex C plane. The domain in this C plane can be restricted by the semicircle theorem (3.8). The eigenvalue C is determined by the value which satisfies the matching relation $U_+(0)\,U'_-(0) - U'_+(0)\,U_-(0) = 0$ and we can easily pick out the eigenvalues from the above plot. This procedure is analogous to that described by Hazel (1972).

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